# PRODUCTS AND FACTORS OF BANACH FUNCTION SPACES

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ABSTRACT. Given two Banach function spaces we study the pointwise product space $E \cdot F$, especially for the case that the pointwise product of their unit balls is again convex. We then give conditions on when the pointwise product $E \cdot M(E, F) = F$, where $M(E, F)$ denotes the space of multiplication operators from $E$ into $F$.

## INTRODUCTION.

Let $(X, \Sigma, \mu)$ be a complete $\sigma$-finite measure space. By $L_0(X, \mu)$ we will denote the set of all measurable functions which are finite a.e.. As usual we will identify functions equal almost evrywhere. A linear subspace of $L_0(X, \mu)$ is called a *Köthe function space* if it is normed space which is an order ideal in $L_0(X, \mu)$, i.e,, if $f \in E$ and $|g| \le |f|$ a.e., then $g \in E$ and $\|g\| \le \|f\|$. A norm complete Köthe function space is called a *Banach function space.* Given a Köthe function space $E$ we have that

$$E' = \{f \in L_0 : \int_X |fg| \, d\mu < \infty \text{ for all } g \in E\}$$

is a Banach function space with the Fatou property, which is called the associate space of $E$. For a detailed treatment of Banach function spaces we refer to [22]. The detailed study of Banach function spaces led to the the study of Riesz spaces and Banach lattices, which incorporated, clarified and extended the earlier theory, see e.g. [23]. In this paper we wil study the pointwise product $E \cdot F = \{f \cdot g : f \in E, g \in F\}$ of two Banach function spaces $E$ and $F$. The main question we will be interested in is whether $E \cdot F$ is again a Banach function space. The finite dimensional case, i.e. the case that $X$ consists of finitely many atoms, shows that one must impose some additional requirements to make this an interesting question. In this paper we will be requiring that the pointwise product $B_E \cdot B_F$ of the respective unit balls of $E$ and $F$ is again the unit ball of a Banach function space, in which case we will call $E \cdot F$ a product Banach function space. In fact, one of the motivations of this paper was a result in a paper of Bollobas and Leader in [4] (see also [5]), who studied the pointwise product of unconditional convex bodies in $\mathbb{R}^n$, which can viewed as a finite dimensional version of one of our results. The main examples of product Banach function spaces are given on the one hand by the fundamental result of Lozanovskii ([13], see also [9]) which states that for any Banach function space $E$ the product $E \cdot E'$ is a product Banach function space isometrically equal to $L_1(X, \mu)$, and on the other hand by the Calderon intermediate space $E^{\frac{1}{p}} F^{\frac{1}{p'}}$ which is a product Banach function space for any pair of Banach function spaces

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and $1 < p < \infty$. In some sense the present paper can be viewed as providing the tools to show that most (if not all) examples in the literature of product Banach function spaces can be deduced from one of these two examples of product Banach function spaces. The present paper is organized as follows. In section 1 we collect some, mostly elementary, results about the normability of the pointwise product of Banach function spaces and show that this is equivalent to having sufficiently many multiplication operators from $E$ into $F'$. This leads us to consider, for any pair $E$ and $F$ of Banach function spaces, the Banach function space of multiplication operators from $E$ into $F$ which we denote by $M(E, F)$, i.e.,

$$M(E, F) = \{g \in L_0(X, \mu) : fg \in F \text{ for all } f \in E\},$$

and the norm on $M(E, F)$ is the operator norm

$$\|g\|_{M(E,F)} = \sup\{\|fg\|_F : \|f\|_E \leq 1\}.$$

We give for non-atomic measures a result which shows that $M(E, F) = \{0\}$ in case the upper index $\sigma(E)$ is less than the lower index $s(F)$ of $F$. This generalizes the well-known result that $M(L^p, L^q) = \{0\}$, whenever $1 \leq p < q \leq \infty$ and $\mu$ is non-atomic. In section 2 we study the basic properties of product Banach function spaces. As the reader will see these results depend heavily on the above mentioned factorization theorem of Lozanovskii. One of the most important results is a cancellation result which says that if $E, F$ and $G$ are Banach function spaces with the Fatou property, and if we assume that $E \cdot F$ and $E \cdot G$ are product Banach function spaces such that $E \cdot F \subset E \cdot G$ with $\|h\|_{E \cdot G} \leq C\|h\|_{E \cdot F}$ for all $h \in E \cdot F$, then $F \subset G$ and $\|f\|_G \leq C\|f\|_F$ for all $f \in F$. In section 3 we consider the problem of division. We will call the Banach function space $E$ a factor of the Banach function space $G$ if there exist a Banach function space $F$ such that $E \cdot F = G$. Lozanovskii's factorization theorem says that every Banach function space is a factor of $L_1(X, \mu)$. To get uniqueness of factors this leads to the question whether $E \cdot M(E, G)$ is a product Banach function space and whether it is equal to $G$. One of the main results we prove is that if $E$ and $G$ are Banach function spaces such that there exists $1 < p < \infty$ such that $E$ is $p$-convex and $G$ is $p$-concave with convexity and concavity constants equal to 1 and if $E$ has the Fatou property, then $] E \cdot M(E, F)$ is a product Banach function space and $E \cdot M(E, F) = F$ and $E = M(M(E, F), F)$. In section 4 we presnt some application of our results. In [1] G. Bennett showed that many classical inequalities involving the $\ell_p$-norm can be expressed as a result of product Banach sequence spaces. We will show that some of his results are easy consequences of the results of the previous sections.

The results of this paper and of [20] were announced at the V-th Positivity conference in July 2007 in Belfast. In March 2008 we learned about the preprint [6], where some of the results of this paper are duplicated independently of this paper.

## 1. The normed product space

Let $E$ and $F$ be Banach function spaces on $(X, \Sigma, \mu)$. We will assume that both $E$ and $F$ are saturated Banach function spaces. In this section we discuss when $E \cdot F = \{f \cdot g : f \in E, g \in F\}$ is a normed Köthe space. The norm, if it exists, will always assumed to be generated by the convex hull of the pointwise product of the

unit balls $B_E \cdot B_F$. In this case one can easily verify that for all $h \in E \cdot F$ we have

$$\|h\|_{E \cdot F} = \inf\left\{\sum_{k=1}^n \|f_k\|_E \|g_k\|_F : |h| \le \sum_{k=1}^n f_k g_k, 0 \le f_k \in E, 0 \le g_k \in F\right\}.$$

In case the above expression defines a norm we will say that $E \cdot F$ is normable. From the discussion in [21] and the fact that $(E \cdot F)^{)\prime}$ contains a strictly positive element in case $E \cdot F$ is normable, we get immediately the following proposition.

**Proposition 1.1.** *Let $E$ and $F$ be Banach function spaces on $(X, \Sigma, \mu)$. Then the following are equivalent.*

(i) $E \cdot F$ *is normable.*
(ii) *There exists $0 < g \in L_0$ such that $g \cdot F \subset E'$.*
(iii) *There exists $0 < g \in L_0$ such that $g \cdot E \subset F'$.*
(iv) *There exist disjoint measurable sets $X_n$ with $\cup_n X_n = X$ such that $F_{|X_n} \subset E'$ for all $n$.*
(v) *There exist disjoint measurable sets $X_n$ with $\cup_n X_n = X$ such that $E_{|X_n} \subset F'$ for all $n$.*

As condition (iv) (or (v)) holds automatically when the measure $\mu$ is atomic (we can take $X_n$ to be an atom for all $n$), we have the following corollary.

**Corollary 1.2.** *Let $E$ and $F$ be Banach function spaces on $(X, \Sigma, \mu)$ and assume $\mu$ is an atomic measure. Then $E \cdot F$ is normable.*

In general it is not true that $E \cdot F$ is complete, whenever $E \cdot F$ is normable, as can be seen from the following simple example.

*Example* 1.3. Let $E = F = \ell_1$. Then $\ell_1 \cdot \ell_1$ as a set is equal to $\ell_{\frac{1}{2}}$, but $\|\cdot\|_{E \cdot F} = \|\cdot\|_1$, so that $(E \cdot F, \|\cdot\|_{E \cdot F})$ is not complete.

In this example and more generally, when $E \cdot F$ is normable, then the completion of $(E \cdot F, \|\cdot\|_{E \cdot F})$ is the Banach envelope of the quasi-normed space $E \cdot F$, provided with the quasi-norm

$$\inf\{\|g\|_E \|h\|_F : |f| = gh, 0 \le g \in E, 0 \le h \in F\}.$$

We refer to [11] for some general remarks about the Banach envelope of locally bounded space with separating dual. For any pair $E$ and $F$ of Banach function spaces we denote by $M(E, F)$ the Banach function space of multiplication operators from $E$ into $F$, i.e.,

$$M(E, F) = \{g \in L_0(X, \mu) : fg \in F \text{ for all } f \in E\},$$

and the norm on $M(E, F)$ is the operator norm

$$\|g\|_{M(E,F)} = \sup\{\|fg\|_F : \|f\|_E \le 1\}.$$

Note that it can happen (as we will see in more detail later on), that $M(E, F) = \{0\}$. The following proposition relates the normability of $E \cdot F$ to the saturation of $M(E, F')$.

**Proposition 1.4.** *Let $E$ and $F$ be Banach function spaces on $(X, \Sigma, \mu)$. Then the following are equivalent.*

(i) $E \cdot F$ *is normable.*
(ii) *The Banach function space $M(E, F')$ is saturated.*

*In case $E \cdot F$ is normable, then $(E \cdot F)' = M(E, F') = M(F, E')$ isometrically.*

*Proof.* The equivalence of (i) and (ii) is just a rephrasing of the equivalence of (i) and (iii) of the above proposition. Assume now that $E \cdot F$ is normable and let $0 \leq h \in (E \cdot f)'$ with $\|h\|_{(E \cdot F)'} \leq 1$. Then let $0 \leq f \in E$ and $0 \leq g \in F$. Then $fg \in E \cdot F$ with $\|fg\|_{E \cdot F} \leq \|f\|_E \|g\|_F$. Hence $0 \leq \int (hf)g \, d\mu \leq \|f\|_E \|g\|_F$. This implies that $hf \in F'$ and $\|hf\|_{F'} \leq \|f\|_E$, i.e., $h \in M(E, F')$ and $\|h\|_{M(E,F)} \leq 1$. This shows $(E \cdot F)' \subset M(E, F')$ and $\|h\|_{M(E,F)} \leq \|h\|_{(E \cdot F)'}$. Now let $0 \leq h \in M(E, F')$ with $\|h\|_{M(E,F')} \leq 1$ and let $0 \leq \tilde{h} \in E \cdot F$ with $\|\tilde{h}\|_{E \cdot F} < 1$. Then there exist $0 \leq f_i \in E$ and $g_i \in F$ with $0 \leq \tilde{h} \leq \sum_{i=1}^{n} f_i g_i$ and $\sum_{i=1}^{n} \|f_i\|_E \|g_i\|_F < 1$. This implies

$$\int \tilde{h}h \, d\mu \leq \sum_{i=1}^{n} \int (hf_i)g_i \, d\mu \leq \sum_{i=1}^{n} \|hf_i\|_{F'} \|g_i\|_F \leq \sum_{i=1}^{n} \|f_i\|_E \|g_i\|_F < 1.$$

Hence $h \in (E \cdot F)'$ and $\|h\|_{(E \cdot F)'} \leq 1$. This shows $M(E, F') \subset (E \cdot F)'$ and $\|h\|_{(E \cdot F)'} \leq \|h\|_{M(E,F)}$, which completes the proof that $(E \cdot F)' = M(E, F')$. Since $E \cdot F = F \cdot E$, it also follows that $(E \cdot F)' = M(F, E')$. $\qquad \square$

**Corollary 1.5.** *Let $E$ be a Banach function space on $(X, \Sigma, \mu)$. Then $E \cdot E$ is normable if and only if there exists $0 < h \in L_0$ such that $E \subset L_2(X, hd\mu)$.*

*Proof.* If $E \cdot E$ is normable, then $(E \cdot E)' = M(E, E')$ contains a strictly positive $h$. Then $\int f^2 hd\mu = \int f(fh) \, d\mu \leq \|f\|_E \|fh\|_{E'} < \infty$ for all $f \in E$, i.e., $E \subset L_2(X, hd\mu)$. Conversely, if $E \subset L_2(X, hd\mu)$ for some strictly positive $h$, then by Cauchy-Schwarz's inequality we have that $\int |f_1 f_2| h \, d\mu < \infty$ for all $f_1, f_2 \in E$, which shows that $hE \subset E'$, so that $E \cdot E$ is normable. $\qquad \square$

We will now show that in certain cases we have $M(E, F) = \{0\}$. First we state a simple lemma.

**Lemma 1.6.** *Let $a, b, c, d, e, f$ be positive real numbers, such that $a \leq b + c$ and $d \geq e + f$. Then*

$$\frac{a}{d} \leq \max \left\{ \frac{b}{e}, \frac{c}{f} \right\}.$$

*Proof.* From the assumption it follows immediately that $\frac{a}{d} \leq \frac{b+c}{e+f}$. Assume $\frac{c}{f} \leq \frac{b}{e}$. Then $\frac{b+c}{e+f} \leq \frac{b}{e}$. Hence $\frac{a}{d} \leq \frac{b+c}{e+f} \leq \frac{b}{e}$. $\qquad \square$

Recall now that a Banach lattice $E$ Similarly a Banach lattice $E$ is called $p$–convex for $1 \leq p \leq \infty$ if there exists a constant $M$ such that for all $f_1, \ldots, f_n \in E$

$$\left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{\frac{1}{p}} \right\|_E \leq M \left( \sum_{k=1}^{n} \|f_k\|_E^p \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty$$

or $\|(\sup |f_k|\|_E \leq M \max_{1 \leq k \leq n} \|f_k\|_E$ if $p = \infty$. Similarly $E$ is called $p$–concave for $1 \leq p \leq \infty$ if there exists a constant $M$ such that for all $f_1, \ldots, f_n \in E$

$$\left( \sum_{k=1}^{n} \|f_k\|_E^p \right)^{\frac{1}{p}} \leq M \left\| \left( \sum_{k=1}^{n} |f_k|^p \right)^{\frac{1}{p}} \right\|_E \quad \text{if } 1 \leq p < \infty$$

or $\max_{1 \leq k \leq n} \|f_k\|_E \leq M \| \sup |f_k| \|_E$ if $p = \infty$. The notions of $p$–convexity, respectively $p$–concavity are closely related to the notions of upper $p$–estimate (strong

$\ell_p$–composition property), respectively lower $p$–estimate (strong $\ell_p$–decomposition property) as can be found in e.g. [12, Theorem 1.f.7]. Then the numbers

$$\sigma(E) = \inf(p \geq 1 : E \text{ is } p\text{-concave })$$

and

$$s(E) = \sup(p \geq 1 : E \text{ is } p\text{-convex })$$

are called the **upper** or **lower index** of $E$, respectively. If $\dim(E) = \infty$, then $1 \leq s(E) \leq \sigma(E) \leq \infty$. We collect some basic facts about the indices of a Banach lattice: If $\sigma(E) < \infty$, then $E$ has order continuous norm and if $s(E) > 1$, then the dual space $E^*$ has order continuous norm. Also we have

$$\frac{1}{s(E)} + \frac{1}{\sigma(E^*)} = 1 \text{ and } \frac{1}{\sigma(E)} + \frac{1}{s(E^*)} = 1.$$

In case $E$ is a Banach function space with the (weak) Fatou property, then we have $s(E^*) = s(E')$ and $\sigma(E^*) = \sigma(E')$. For this and additional details see [10] and [8].

**Lemma 1.7.** *Let $E$ be a Banach function space on $(X, \Sigma, \mu)$. Assume $E$ has order continuous norm and $\mu$ is non-atomic. Then for all $0 \leq f \in E$ there exist $0 \leq g \leq f \in E$ with $g \wedge (f - g) = 0$ such that $\|g\|_E = \|f - g\|_E$.*

*Proof.* First observe that for all $0 \leq f \in E$ with $f \neq 0$ we can find for all $\epsilon > 0$ a component $0 \leq g \leq f \in E$ with $g \wedge (f - g) = 0$ such that $0 < \|g\|_E < \epsilon$. In fact we can find a sequence $X_n \downarrow \emptyset$ with $0 < \mu(X_n) < \frac{1}{n}$. Then $\|f\chi_{X_n}\|_E \downarrow 0$ implies that there exists $n_0$ such that $\|f\chi_{X_{n_0}}\|_E < \epsilon$. Now let $0 \leq f \in E$ and consider the set $\mathcal{P} = \{g \in E : 0 \leq g \leq f, g \wedge (f - g) = 0, \|g\|_E \leq \|f - g\|_E\}$. Then $\mathcal{P} \neq \emptyset$ and with the ordering inherited from $E$ it has the property that every chain in $\mathcal{P}$ has a least upper bound, by the order continuity of the norm. Hence $\mathcal{P}$ has a maximal element $g_0$. Assume $\|g_0\|_E < \|f - g_0\|_E$. Then take $\epsilon = \frac{1}{2}(\|f - g_0\|_E - \|g_0\|_E)$. By the remark in the beginning of the proof we can a component $0 < g_1 \leq (f - g_0)$ of $f - g_0$ such that $\|g_1\|_E < \epsilon$. One can verify now easily that $g_0 + g_1 \in \mathcal{P}$, which contradicts the maximality of $g_0$. Hence $\|g_0\|_E = \|f - g_0\|_E$ and the proof is complete. $\square$

**Theorem 1.8.** *Let $E$ and $F$ be Banach function spaces on $(X, \Sigma, \mu)$ and assume $\mu$ is non-atomic. Then $\sigma(E) < s(F)$ implies that $M(E, F) = \{0\}$.*

*Proof.* Let $\sigma(E) < p < q < s(F)$. Then $E$ is $p$-concave and $F$ is $q$-convex. Renorming $E$ and $F$, if necessary, we can assume that the concavity and convexity constants are 1. Also $p < \infty$ implies that the space $E$ has order continuous norm. Let now $0 \leq h \in M(E, F)$ and $0 \leq f_0 \in E$ with $\|f_0\|_E > 0$. By the above lemma we can write $f_0 = g_0 + h_0$ with $g_0 \wedge h_0 = 0$ and such that $\|g_0\|_E = \|h_0\|_E$. By the p-concavity we have $\|f\|_E^p \geq \|g_0\|_E^p + \|h_0\|_E^p = 2\|g_0\|_p$ and by the $q$-convexity of $F$ we have $\|hf_0\|_F^q \leq \|hg_0\|_F^q + \|hh_0\|_F^q$. It follows now from Lemma 1.6 that there exists $f_1$ equal to either $g_0$ or $h_0$ such that

$$\frac{\|hf_1\|_F^q}{\|f_1\|_E^p} \geq \frac{\|hf_0\|_F^q}{\|f_0\|_E^p}.$$

By induction we can now find $f_{n+1} \geq 0$ such that $\|f_n\|_E^p \geq 2\|f_{n+1}\|^p$ and

$$\frac{\|hf_{n+1}\|_F^q}{\|f_{n+1}\|_E^p} \geq \frac{\|hf_n\|_F^q}{\|f_n\|_E^p}.$$

Assume now that $hf_0 \neq 0$. Then we have

$$\|h\|_{M(E,F)}^q \geq \frac{\|hf_n\|_F^q}{\|f_n\|_E^q} = \frac{\|hf_n\|_F^q}{\|f_n\|_E^p} \cdot \frac{1}{\|f_n\|_E^{q-p}}$$

$$\geq \frac{\|hf_0\|_F^q}{\|f_0\|_E^q} \cdot 2^{\frac{n(q-p)}{p}} \cdot \frac{1}{\|f_0\|_E^{q-p}} \to \infty,$$

which is contradiction. Hence $hf_0 = 0$ for all $f_0 \in E$ and thus $h = 0$. $\qquad\square$

**Corollary 1.9.** *Let $E$ and $F$ be Banach function spaces on $(X, \Sigma, \mu)$ and assume $\mu$ is non-atomic. Then $\frac{1}{\sigma(E)} + \frac{1}{\sigma(F)} > 1$ implies that $E \cdot F$ is not normable.*

*Proof.* From $\frac{1}{\sigma(E)} + \frac{1}{\sigma(F)} > 1$, it follows that $\frac{1}{\sigma(E)} > 1 - \frac{1}{\sigma(F)} = \frac{1}{s(F')}$. Hence by the above proposition $M(E, F') = \{0\}$. $\qquad\square$

## 2. Product Banach function spaces

As seen from the examples in the previous section it can happen that $E \cdot F$ is a normed Köthe space, but is not complete. To better understand when $E \cdot F$ is a Banach function space we need the Calderon construction of intermediate spaces, which was studied and extended by Lozanovskii. Let $E$ and $F$ be Banach function spaces on $(X, \mu)$. Then for $1 < p < \infty$ the Banach function space $E^{\frac{1}{p}} F'^{\frac{1}{p'}}$ is defined as the space of all $f \in L_0$ such that $|f| = |g|^{\frac{1}{p}} |h|^{\frac{1}{p'}}$ for some $g \in E$ and $h \in F$. The norm on $E^{\frac{1}{p}} F'^{\frac{1}{p'}}$ is defined by

$$\|f\|_{E^{\frac{1}{p}} F^{\frac{1}{p'}}} = \inf\{\|g\|_E^{\frac{1}{p}} \|h\|_F^{\frac{1}{p'}} : |f| = |g|^{\frac{1}{p}} |h|^{\frac{1}{p'}} \text{ for some } g \in E, h \in F\}$$

$$= \inf\{\lambda : |f| = \lambda |g|^{\frac{1}{p}} |h|^{\frac{1}{p'}}, \|g\|_E \leq 1, \|h\|_F \leq 1\}.$$

It is well-known that $E^{\frac{1}{p}} F'^{\frac{1}{p'}}$ is again a Banach function space, moreover it has order continuous norm if at least one of $E$ and $F$ has order continuous norm. Also $E^{\frac{1}{p}} F'^{\frac{1}{p'}}$ has the Fatou property if both $E$ and $F$ have the Fatou property. The above construction contains as a special case the so-called $p$-convexication of $E$ by taking $F = L_\infty$. We will denote this space by $E^{\frac{1}{p}}$, since as sets $E^{\frac{1}{p}} L_\infty^{\frac{1}{p'}} = E^{\frac{1}{p}}$. Note that the norm on $E^{\frac{1}{p}}$ is given by $\||f|^p\|_E^{\frac{1}{p}}$ for all $f \in E^{\frac{1}{p}}$. This implies that the space $E^{\frac{1}{p}} F'^{\frac{1}{p'}} = E^{\frac{1}{p}} \cdot F'^{\frac{1}{p'}}$ as defined in the previous section and that

$$\|f\|_{E^{\frac{1}{p}} \cdot F^{\frac{1}{p'}}} = \inf\{\|g\|_{E^{\frac{1}{p}}} \|h\|_{F^{\frac{1}{p'}}} : |f| = |g| \cdot |h|, g \in E^{\frac{1}{p}}, h \in F'^{\frac{1}{p'}}\}.$$

In particular the pointwise product of the unit balls $B_{E^{\frac{1}{p}}}$ and $B_{F^{\frac{1}{p'}}}$ is convex. Let again $E$ and $F$ be Banach function spaces on $(X, \Sigma, \mu)$. Assume that $E \cdot F$ is a normable Köthe function space. Then we will say that $E \cdot F$ is a *product Banach function space* if $E \cdot F$ is complete and the norm on $E \cdot F$ is given by

$$\|f\|_{E \cdot F} = \inf\{\|g\|_E \|h\|_F : |f| = gh, 0 \leq g \in E, 0 \leq h \in F\},$$

i.e., the pointwise product $B_E \cdot B_F$ of the unit balls of $E$, respectively $F$, is convex. From the above discussion it is clear that $E^{\frac{1}{p}} \cdot F'^{\frac{1}{p'}}$ is a product Banach function space. Also the fundamental result of Lozanovskii ([13], see also [9]) is that for any Banach function space $E$ the product $E \cdot E'$ is a product Banach function space isometrically equal to $L_1(X, \mu)$. We first show that completeness can be omitted from our definition of product Banach function space. Then we will recall from [20]

that more generally the pointwise product $B_E \cdot B_F$ of the unit balls $B_E$ and $B_F$ is closed with respect to a.e. convergence in case the norms on $E$ and $F$ have the Fatou property. Recall that if $E \cdot F$ is a normable Köthe space, then for $f \in E \cdot F$ the quasi-norm of $f \in E \cdot F$ is given by

$$\rho_{E \cdot F}(f) = \inf\{\|g\|_E\|h\|_F : |f| = gh, 0 \le g \in E, 0 \le h \in H\}.$$

Note that $\|f\|_{E \cdot F}(f) \le \rho_{E \cdot F}(f)$ and equality holds if and only if $B_E \cdot B_F$ is convex. In the next theorem we will need the $p$-concavification of a Banach function space. If $E$ is $p$-convex for some $p > 1$, then we can define the $p$-concavification $E^p$ by $f \in E^p$ if $f \in L_0(X, \mu)$ such that $|f|^p \in E$. If the convexity constant is equal to one, then $\|f\|_{E^p} = \| |f|^p \|_E^{\frac{1}{p}}$ is a norm on $E^p$ and $E^p$ is complete with respect to this norm (see [12]).

**Theorem 2.1.** *Let $E$ and $F$ be Banach function spaces and assume that $\rho_{E \cdot F}$ is a norm. Then $E \cdot F$ is complete with respect to $\rho_{E \cdot F}$, i.e. $E \cdot F$ is a product Banach function space.*

*Proof.* Let $G = E^{\frac{1}{2}} F^{\frac{1}{2}}$. Then $G$ is a Banach function space and $G$ is the 2-convexification of the normed lattice $E \cdot F$ with the norm $\rho_{E \cdot F}$. In particular $G$ is 2-convex with convexity constant equal to one. Now $E \cdot F$ is isometric to the 2-concavification of $G$ and thus $E \cdot F$ is complete.                        □

**Corollary 2.2.** *Let $E$ and $F$ be Banach function spaces. Then $E \cdot F$ is a product Banach function space if and only if $E^{\frac{1}{2}} F^{\frac{1}{2}}$ is 2-convex with convexity constant one. In particular $E \cdot E$ is a product Banach function space if and only if $E$ is 2-convex with convexity constant one.*

From [20] we have the following result.

**Theorem 2.3.** *Let $E$ and $F$ be Banach function spaces with the Fatou property. Then $B_E \cdot B_F$ is closed in $L_0(X, \mu)$ with respect to a.e. convergence. If addition $\rho_{E \cdot F}$ is a norm, then $E \cdot F$ is a product Banach function space with the Fatou property.*

The following theorem says for product Banach function spaces defined by Banach function spaces withe the Fatou property that the infimum in the definition of the norm is actually attained.

**Theorem 2.4.** *Let $E$ and $F$ be Banach function spaces with the Fatou property and assume that $E \cdot F$ is a product Banach function space. Then for all $0 \le f \in E \cdot F$ there exist $0 \le g \in E$ and $0 \le h \in F$ such that $f = gh$ and $\|f\|_{E \cdot F} = \|g\|_E\|h\|_F$.*

*Proof.* Let $0 \le f \in E \cdot F$ with $\|f\|_{E \cdot F} = 1$. Then there exist $0 \le g_n \in E$, $0 \le h_n \in F$ with $f = g_n h_n$ and $\|g_n\|_E \le 1 + \frac{1}{2^n}$, $\|h_n\|_F \le 1 + \frac{1}{2^n}$ for all $n \ge 1$. From Komlós' theorem for Banach function spaces and a theorem on products of Cesaro convergent sequences (see [20]) it follows that there exist subsequences $\{g_{n_k}\}$ and $\{h_{n_k}\}$ and $g \in E$ with $\{g_{n_k}\}$ Cesàro converges a.e to $g$ and $\{h_{n_k}\}$ Cesàro converges a.e to $h$ such that $f \le gh$. Replacing $h$ by a smaller function we can assume $f = gh$. Moreover

$$\|\frac{1}{k}(g_{n_1} + \cdots g_{n_k})\|_E \le \frac{1}{k}(1 + \frac{1}{2^{n_1}} + \cdots + 1 + \frac{1}{2^{n_k}}) \le 1 + \frac{1}{k}$$

implies that $\|g\|_E \le 1$. Similarly $\|h\|_F \le 1$. As $\|f\|_{E \cdot F} \le \|g\|_E\|h\|_F$ this implies that $\|g\|_E = 1$ and $\|h\|_F = 1$.

$\square$

**Theorem 2.5.** *Let $E, F$ and $G$ be Banach function spaces with the Fatou property. Assume that $E \cdot F$ and $E \cdot G$ are product Banach function spaces such that $E \cdot F \subset E \cdot G$ with $\|h\|_{E \cdot G} \leq C \|h\|_{E \cdot F}$ for all $h \in E \cdot F$. Then $F \subset G$ and $\|f\|_G \leq C \|f\|_F$ for all $f \in F$.*

*Proof.* If $0 \leq f \in F$, then there exist $0 \leq f_n \uparrow f$ such that $f_n \in F \cap G$. Hence if $\|f\|_G \leq C \|f\|_F$ for all $f \in F \cap G$, then the same inequality holds by the Fatou property for all $f \in F$. Assume therefore that there exists $0 < f \in F \cap G$ such that $\|f\|_G > C \|f\|_F$. By normalizing we can assume that $\|f\|_F = 1$ and thus $\|f\|_G > C$. Then there exists $0 \leq g \in G'$ such that $C_1 = \int fg \, d\mu > C$ and $\|g\|_{G'} \leq 1$. Now $\frac{1}{C_1} fg \in L^1$ with $\|\frac{1}{C_1} fg\|_1 = 1$, so by Lozanovskii's theorem there exist $0 \leq f_1 \in E$ with $\|f_1\|_E \leq 1$ and $0 \leq g_1 \in E'$ with $\|g_1\|_{E'} \leq 1$ such that $\frac{1}{C_1} fg = f_1 g_1$. It follows now that

$$\int f^{\frac{1}{2}} g^{\frac{1}{2}} f_1^{\frac{1}{2}} g_1^{\frac{1}{2}} \, d\mu = \frac{1}{C_1^{\frac{1}{2}}} \int fg \, d\mu = C_1^{\frac{1}{2}}.$$

On the other hand $f_1 f \in E \cdot F$ with $\|f_1 f\|_{E \cdot F} \leq \|f_1\|_E \|f\|_F \leq 1$ implies that $\|f_1 f\|_{E \cdot G} \leq C$. Hence for $C < C_2 < C_1$ there exist $f_2 \in E$ with $\|f_2\|_E \leq 1$ and $g_2 \in G$ with $\|g_2\|_G \leq C_2$ such that $f_1 f = f_2 g_2$. This implies that

$$\int f^{\frac{1}{2}} g^{\frac{1}{2}} f_1^{\frac{1}{2}} g_1^{\frac{1}{2}} \, d\mu = \int g_2^{\frac{1}{2}} g^{\frac{1}{2}} g_1^{\frac{1}{2}} f_2^{\frac{1}{2}} \, d\mu \leq \left( \int g_2 g \, d\mu \right)^{\frac{1}{2}} \left( \int g_1 f_2 \, d\mu \right)^{\frac{1}{2}}$$

$$\leq \|g_2\|_G^{\frac{1}{2}} \|g\|_{G'}^{\frac{1}{2}} \|g_1\|_{E'}^{\frac{1}{2}} \|f_2\|_E^{\frac{1}{2}} \leq C_2^{\frac{1}{2}} < C_1^{\frac{1}{2}},$$

which is a contradiction. $\square$

**Corollary 2.6.** *Let $E, F$ and $G$ be Banach function spaces with the Fatou property. Assume that $E \cdot F$ and $E \cdot G$ are product Banach function spaces with $E \cdot F = E \cdot G$ isomorphically (or isometrically). Then $F = G$ isomorphically (or isometrically).*

Note that the above corollary is no longer true if we drop the assumption of the Fatou property, we have e.g. that $\ell_1 \cdot c_0 = \ell_1 \cdot \ell_\infty = \ell_1$. However the same proofs would give that $F'' \subset G''$ in the above theorem and $F'' = G''$ in the above corollary, if we omit the hypothesis of the Fatou property for the spaces $F$ and $G$. Moreover in the above theorem and corollary we can omit the hypothesis that $E$ has the Fatou property by using Lozanovskii's approximate factorization (this introduces a $1 + \epsilon$ term in the factorization). The proof of the above theorem was inspired by a proof of [2], who used it to give an alternative proof of the uniqueness theorem of the Calderon-Lozaonovskii interpolation method as given in [7]. The following proposition provides a strengthening of the above corollary and seems to be new even for Calderon spaces.

**Proposition 2.7.** *Let $E_1, E_2, F_1$ and $F_1$ be Banach function spaces with the Fatou property. Assume that $E_1 \cdot F_1$ and $E_2 \cdot F_2$ are product Banach function spaces such that $E_1 \cdot F_1 = E_2 \cdot F_2$ isomorphically (or isometrically) and $E_1 \subset E_2, F_1 \subset F_2$ (or additionally with norm 1 inclusions). Then $E_1 = E_2$ and $F_1 = F_2$ isomorphically (or isometrically).*

*Proof.* Observe that $E_2^{\frac{1}{2}} \cdot F_1^{\frac{1}{2}} \supset E_1^{\frac{1}{2}} \cdot F_1^{\frac{1}{2}} = E_2^{\frac{1}{2}} \cdot F_2^{\frac{1}{2}}$ implies by the above corollary $F_1^{\frac{1}{2}} \supset F_2^{\frac{1}{2}}$. Hence $F_1 \supset F_2$. This implies that $F_1$ is isomorphic (or isometric) to $F_2$. Similarly the conclusion holds for $E_1$ and $E_2$, which completes the proof. $\square$

We will now show how the above results can be used to derive an alternative proof of Lozanovskii's duality theorem (see [13], [14], [16], [18], and [19] for additional information concerning this duality in the more general case) for Calderon product spaces of Banach function spaces with the Fatou property.

**Theorem 2.8.** *Let $E$ and $F$ be Banach function spaces with the Fatou property. Assume that $E \cdot F$ is a product Banach function space. Then $F = M(E, E \cdot F)$ isometrically.*

*Proof.* Let $f \in B_E$ and $g \in F$. Then $\|fg\|_{E \cdot F} \leq \|f\|_E \|g\|_F \leq \|g\|_F$ shows that $F \subset M(E, E \cdot F)$ and $B_F \subset B_{M(E,F)}$. Hence $B_E \cdot B_F \subset B_E \cdot B_{M(E,F)}$. On the other hand by the definition of the norm of $M(E, E \cdot F)$ we have that $B_E \cdot B_{M(E,E \cdot F)} \subset B_{E \cdot F} = B_E \cdot B_F$. Hence $B_E \cdot B_F = B_E \cdot B_{M(E,E \cdot F)}$. This shows that $E \cdot M(E, E \cdot F)$ is a product Banach function space isometric to $E \cdot F$. From the above corollary we conclude that $F = M(E, E \cdot F)$ isometrically. $\square$

We note that if $E$ is $p$-convex and $F$ is $p'$-convex, then the above theorem reproves Theorem 3.5 of [7], which was the main tool used there to prove the uniqueness theorem of the Calderon-Lozaonovskii interpolation method. The next theorem is a special case of Lozanovskii's duality theorem.

**Theorem 2.9.** *Let $F$ be a Banach function space and let $1 < p < \infty$. Then $(F^{\frac{1}{p}})' = (F')^{\frac{1}{p}} L_1^{\frac{1}{p'}} = (F')^{\frac{1}{p}} \cdot L_{p'}$.*

*Proof.* From Lozanovskii's factorization theorem we have isometrically $F^{\frac{1}{p}} (F^{\frac{1}{p}})' = L_1$. On the other hand $F^{\frac{1}{p}} \left( (F')^{\frac{1}{p}} \cdot L_{p'} \right) = (F \cdot F')^{\frac{1}{p}} \cdot L_{p'} = L_1^{\frac{1}{p}} \cdot L_{p'} = L_1$ isometrically. Hence by the above corollary $(F^{\frac{1}{p}})' = (F')^{\frac{1}{p}} \cdot L_{p'}$ isometrically. Note that both spaces $(F^{\frac{1}{p}})'$ and $(F')^{\frac{1}{p}} \cdot L_{p'}$ have the Fatou property, and by the remark above we do not need the Fatou property for $E$. $\square$

In the following theorem we provide a simple proof of Lozanovskii's duality theorem for the case of Banach function spaces with the Fatou property.

**Theorem 2.10.** *Let $E$ and $F$ be Banach function spaces with the Fatou property and let $1 < p < \infty$. Then*

$$\left( E^{\frac{1}{p}} F^{\frac{1}{p'}} \right)' = (E')^{\frac{1}{p}} (F')^{\frac{1}{p'}}.$$

*Proof.* From Proposition 1.4 it follows that $\left( E^{\frac{1}{p}} F^{\frac{1}{p'}} \right)' = M(E^{\frac{1}{p}}, (F^{\frac{1}{p'}})')$, which by the above theorem is equal to $M(E^{\frac{1}{p}}, (F')^{\frac{1}{p'}} \cdot L_p)$. Now $M(E^{\frac{1}{p}}, (F')^{\frac{1}{p'}} \cdot L_p) = M(E^{\frac{1}{p}}, E^{\frac{1}{p}} (E')^{\frac{1}{p}} \cdot (F')^{\frac{1}{p'}}) = M(E^{\frac{1}{p}}, E^{\frac{1}{p}} \cdot (E')^{\frac{1}{p}} (F')^{\frac{1}{p'}}) = (E')^{\frac{1}{p}} (F')^{\frac{1}{p'}}$ by Theorem 2.8. $\square$

**Corollary 2.11.** *Let $F$ be a Banach function space and let $1 < p < \infty$. Then $(F^{\frac{1}{p}})'' = (F'')^{\frac{1}{p}}$.*

*Proof.* By Theorem 2.9 we have that $(F^{\frac{1}{p}})' = (F')^{\frac{1}{p}} L_1^{\frac{1}{p'}}$. Hence by the above theorem $(F^{\frac{1}{p}})'' = (F'')^{\frac{1}{p}} (L_\infty)^{\frac{1}{p'}} = (F'')^{\frac{1}{p}}$.

$\square$

## 3. Factors of Banach function spaces

Besides considering products of Banach function spaces one can consider the problem of division. We will call the Banach function space $E$ a factor of the Banach function space $G$ if there exist a Banach function space $F$ such that $E \cdot F = G$. Lozanovskii's factorization theorem says that every Banach function space is a factor of $L_1(X, \mu)$. As the example $\ell_1 \cdot c_0 = \ell_1 \cdot \ell_\infty = \ell_1$ shows, the space $F$ is not unique, if no additional requirements are imposed. Now $E \cdot F = G$ implies that $F \subset M(E, G)$ and thus $E \cdot M(E, G) = G$. Therefore it is natural to assume that $F = M(E, G)$ and that the norm on $F$ is given by the operator norm of multiplying $E$ into $G$. We will first determine the factors of $L_p$ and for that reason we first derive some elementary properties of $M(E, L_p)$.

**Proposition 3.1.** *Let $E$ be a Banach function space. Then $M(E, L_p)$ is a $p$-convex Banach function space with convexity constant one.*

*Proof.* Let $0 \le f_k \in M(E, L_p)$ for $1 \le k \le n$ and let $0 \le g \in E$. Then

$$\left\| \left( \sum_{k=1}^{n} f_k^p \right)^{\frac{1}{p}} g \right\|_p = \left\| \left( \sum_{k=1}^{n} (f_k g)^p \right)^{\frac{1}{p}} \right\|_p$$

$$= \left( \sum_{k=1}^{n} \| f_k g \|_p^p \right)^{\frac{1}{p}}$$

$$\le \left( \sum_{k=1}^{n} \| f_k \|_{M(E, L_p)}^p \right)^{\frac{1}{p}} \| g \|_E.$$

Hence $M(E, L_p)$ is $p$-convex with convexity constant one.  $\square$

In Theorem 2.9 we saw a description of the Köthe dual of the $p$-convexification of a Banach function space. In the following theorem we characterize the Köthe dual of the $p$-concavification of a Banach function space by means of the $p$-concavification of the $p$-convex space $M(E, L_p)$.

**Theorem 3.2.** *Let $E$ be a Banach function space and assume $E$ is $p$-convex for some $p > 1$ with convexity constant one. Then the Köthe dual $(E^p)'$ of the $p$-concavification $E^p$ of $E$ is isometric with the $p$-concavification $M(E, L_p)^p$ of $M(E, L_p)$.*

*Proof.* Let $0 \le f \in L_0$. Then we have

$$\| f \|_{(E^p)'} = \sup \left( \int f |g|^p \, d\mu : \| |g|^p \|_{E^p} \le 1 \right)$$

$$= \sup \left( \| |f|^{\frac{1}{p}} |g| \|_p^p : \| g \|_E \le 1 \right)$$

$$= \| f^{\frac{1}{p}} \|_{M(E, L_p)}^p$$

$$= \| f \|_{M(E, L_p)^p}$$

This identity proves the theorem.  $\square$

**Theorem 3.3.** *Let $E$ be a Banach function space with the Fatou property and let $1 < p < \infty$. Then the following are equivalent.*

(i) $E$ is $p$-convex with convexity constant one.
(ii) $E \cdot M(E, L_p)$ is a product Banach function space and $E \cdot M(E, L_p) = L_p$.
(iii) $M(M(E, L_p), L_p) = E$.

*Proof.* Assume first that $E$ is $p$-convex with convexity constant one. Then by the above theorem $(E^p)' = M(E, L_p)^p$. Thus by Lozanovskii's theorem $E^p \cdot M(E, L_p)^p = L_1$. Now taking the $p$-convexication on both sides we see that (ii) holds. Now assume (ii) holds. Then $M(E, L_p)$ is a $p$-convex Banach function space with the Fatou property and $L_{p'}$ is $p'$-convex with the Fatou property, so $M(E, L_p) \cdot L_{p'}$ is a product Banach function space with the Fatou property. Now by (ii) we have that $E \cdot (M(E, L_p) \cdot L_{p'}) = L_p \cdot L_{p'} = L_1$. Hence $E' = M(E, L_p) \cdot L_{p'}$. From this it follows, using Lozanovskii's duality theorem, that

$$
\begin{aligned}
M(M(E, L_p), L_p) &= [(M(E, L_p)^p)']^{\frac{1}{p}} \\
&= [(M(E, L_p)^p)']^{\frac{1}{p}} (L_\infty)^{\frac{1}{p'}} \\
&= (M(E, L_p) \cdot L_{p'})' = E'' = E.
\end{aligned}
$$

Hence (iii) holds. Assume now that (iii) holds. Then $E$ is $p$-convex, since $M(F, L_p)$ is $p$-convex for any Banach function space $F$. $\qquad\square$

Note that the $p = 1$ case of the above theorem is Lozanovskii's factorization theorem, which also shows that as stated we can not drop the assumption that $E$ has the Fatou property. The $p = 1$ case and the above proof suggest however that there might be a version without the Fatou property, if we replace in (iii) the space $E$ on the right hand side by $E''$. We were not able to establish this in general, but one important special case follows.

**Theorem 3.4.** *Let $E$ be a Banach function space with order continuous norm and let $1 < p < \infty$. Then the following are equivalent.*

(i) $E$ is $p$-convex with convexity constant one.
(ii) $E \cdot M(E, L_p)$ is a product Banach function space and $E \cdot M(E, L_p) = L_p$.
(iii) $M(M(E, L_p), L_p) = E''$.

*Proof.* An inspection of the above proof shows that the only step which needs modification is the proof that (iii) implies (i). From (iii) it follows that $E''$ is $p$-convex. This implies that $E$ is $p$-convex, since the order continuity of the norm on $E$ implies that $\|f\|_E = \|f\|_{E''}$ for all $f \in E$. $\qquad\square$

As is clear from the above proof the difficulty in establishing the above theorem without any additional assumption on $E$ is to show that if $E''$ is $p$-convex, then $E$ is $p$-convex. One can observe that if $E''$ is $p$-convex, then $(E'')^*$ is $p'$-concave, so also $E' = E'''$ is $p'$-concave. The difficulty is then to show that this implies that $E$ is $p$-convex without any additional hypotheses.

**Corollary 3.5.** *Let $E$ and $F$ be Banach function spaces. Assume that $F = M(E, L_p)$ and $E = M(F, L_p)$ for some $1 < p < \infty$. Then $E \cdot F$ is a product Banach function space and $E \cdot F = L_p$.*

*Proof.* Since $L_p$ has the Fatou property it follows that $M(E, L_p)$ and $M(F, L_p)$ have the Fatou property. Moreover $E = M(F, L_p)$ implies that $E$ is $p$-convex and the result follows. $\qquad\square$

**Remark.** The above corollary can be rephrased as follows. If $K$ and $L$ are unconditional convex bodies in $L_0$, which are both maximal for the inclusion $K \cdot L \subset B_{L_p}$, then $K \cdot L = B_{L_p}$. This result for unconditional convex bodies in $\mathbb{R}^n$ was proved by Bollobas and Leader in [4] (see also [5]) by a completely different method. We also note that the above Theorem 3.3 provides a partial answer to Question 2 of [15], where it was asked (with a different notation): for which Banach function spaces $E$ and $G$ with $G \neq L_1$ is it true that $M(M(E,G),G) = E$. We will provide a more general answer later on.

One can conjecture more general versions of the above corollary and in fact this was done for finite dimesional unconditional bodies in [4] and [5]. One of such conjectures was answered in the negative in Theorem 6 of [5]. We will present the answer to another such conjecture, Conjecture 7 of [5], which was already answered in the negative in [3]. Our example is essentially the same as the one in [3], but does not use any graph theoretical techniques.

*Example* 3.6. Let $X = \mathbb{R}^3$ with the counting measure. Define $E = \mathbb{R}^3$ with the norm $\|x\|_E = \max\{|x_1| + |x_2|, |x_3|\}$ and $G = \mathbb{R}^3$ with the norm $\|z\|_G = |z_1| + \max\{|z_2|, |z_3|\}$. Let $F = M(E,G)$. To compute $\|y\|_F$ we shall use the well-known fact that a convex function attains its maximum on a compact convex set at an extreme point of the compact convex set.

$$\begin{aligned}
\|y\|_F &= \max\{|x_1y_1| + \max\{|x_2y_2|, |x_3y_3|\} : |x_3| \leq 1, |x_1| + |x_2| \leq 1\} \\
&= \max\{|x_1y_1| + \max\{|x_2y_2|, |y_3|\} : |x_1| + |x_2| \leq 1\} \\
&= \max\{|y_1| + |y_3|, \max\{|y_2|, |y_3|\}\} = \max\{|y_1| + |y_3|, |y_2|\}.
\end{aligned}$$

Noting the symmetry in the norms of $E$ and $F$ we see immediately that $E = M(F,G)$. We shall show now that $B_E \cdot B_F$ is not convex, so that $E \cdot F$ is not a product Banach function space. Observe that $(1,0,0) = (1,0,0) \cdot (1,1,0) \in B_E \cdot B_F$ and $(0,1,1) = (0,1,1) \cdot (0,1,1) \in B_E \cdot B_F$. We claim that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(1,0,0) + \frac{1}{2}(0,1,1) \notin B_E \cdot B_F$, while it is clear that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in B_G$. Assume $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = (x_1y_1, x_2y_2, x_3y_3)$, where $0 \leq (x_1, x_2, x_3) \in B_E$ and $0 \leq (y_1, y_2, y_3) \in B_F$. Then $\|x\|_E \leq 1$ and $\|y\|_F \leq 1$ imply that $x_i \leq 1$ and $y_i \leq 1$ for $i = 1, 2, 3$. Now $x_2y_2 = \frac{1}{2}$ implies that either $x_2 = 1$ and $y_2 = \frac{1}{2}$, or $x_2 = \frac{1}{2}$ and $y_2 = 1$. If $x_2 = 1$, then $x_1 = 0$, contradicting $x_1y_1 = \frac{1}{2}$ and if $x_2 = \frac{1}{2}$, then $x_1 = \frac{1}{2}$. so $y_1 = 1$ which implies $y_3 = 0$. contradicting $x_3y_3 = \frac{1}{2}$. Hence $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin B_E \cdot B_F$. We also observe also that $B_G$ is the convex hull of $B_E \cdot B_F$. To see this one can check very easily that the intersections of $B_E \cdot B_F$ and $B_G$ with each of the coordinate planes coincide. Now the positive part of the ball $B_G$ is the convex hull of the intersections of $B_{E+} \cdot B_{F+}$ with each of the coordinate planes and the line segment connecting $(1,0,0)$ and $(0,1,1)$, so $B_G$ is the convex hull of $B_E \cdot B_G$. We illustrate this by the following picture, which shows part of $B_E \cdot B_F$.

To get a dual version of Theorem 3.3 we present first a more general duality result.

**Theorem 3.7.** *Let $E$ and $F$ be Banach function spaces withe the Fatou property. If $E \cdot F = G$ is a product Banach function space, then $E \cdot G'$ is a product Banach function space and $E \cdot G' = F'$.*

*Proof.* It is easy to see that $G$ is a Banach function space with the Fatou property. Now $E^{\frac{1}{2}} \cdot F^{\frac{1}{2}} = G^{\frac{1}{2}}$ implies that $E'^{\frac{1}{2}}F'^{\frac{1}{2}} = G'^{\frac{1}{2}} \cdot L_2$. Hence $E^{\frac{1}{2}} \cdot E'^{\frac{1}{2}}F'^{\frac{1}{2}} =$

$E^{\frac{1}{2}} \cdot G'^{\frac{1}{2}} \cdot L_2$, which can be written as $L_2 \cdot F'^{\frac{1}{2}} = (E^{\frac{1}{2}} G'^{\frac{1}{2}}) \cdot L_2$. One can check then that $\| \cdot \|_{L_2 \cdot F'^{\frac{1}{2}}} = \rho_{(E^{\frac{1}{2}} G'^{\frac{1}{2}}) \cdot L_2}(\cdot)$, and thus $\rho_{(E^{\frac{1}{2}} G'^{\frac{1}{2}}) \cdot L_2}$ is a norm. Thus $(E^{\frac{1}{2}} G'^{\frac{1}{2}}) \cdot L_2$ is a product Banach function space with the Fatou property. Hence $F'^{\frac{1}{2}} = E^{\frac{1}{2}} G'^{\frac{1}{2}}$. This shows that $E^{\frac{1}{2}} G'^{\frac{1}{2}}$ is 2-convex with convexity constant 1 and thus $E \cdot G'$ is a product Banach function space. From $F'^{\frac{1}{2}} = E^{\frac{1}{2}} G'^{\frac{1}{2}}$ it follows then that $F' = E \cdot G'$.

$\square$

The following theorem gives a more general sufficient condition for $E \cdot M(E, F) = F$ and $E = M(M(E, F), F)$ to hold.

**Theorem 3.8.** *Let $E$ and $F$ be Banach function spaces such that there exists $1 < p < \infty$ such that $E$ is $p$-convex and $F$ is $p$-concave with convexity and concavity constants equal to 1 and assume $E$ has the Fatou property. Then the following hold.*

 (i) *$E \cdot M(E, F)$ is a product Banach function space and $E \cdot M(E, F) = F$.*
 (ii) *$E = M(M(E, F), F)$.*

*Proof.* For the proof of (i) observe first that $F'$ is $p'$-convex, so that $E \cdot F' = (E^p)^{\frac{1}{p}} \left( F'^{p'} \right)^{\frac{1}{p'}}$ is a product Banach function space with the Fatou property. Let $G = E \cdot F'$. Then by Theorem 3.7 we have that $E \cdot G' = F'$. From the results in section 1 we have that $G' = (E \cdot F^{p'})' = M(E, F'') = M(E, F)$. Hence $E \cdot M(E, F) = F$. This shows that (i) holds. To prove (ii) note that by (i) we have $M(E.F) \cdot F' = E'$. Hence $M(M(E, F), F) = (M(E, F) \cdot F')' = E'' = E$. $\square$

**Theorem 3.9.** *Let $F$ be a Banach function space with the Fatou property and let $1 < p < \infty$. Then the following are equivalent.*

 (i) *$F$ is $p$-concave with concavity constant one.*
 (ii) *$M(L_p, F) \cdot L_p$ is a product Banach function space and $M(L_p, F) \cdot L_p = F$.*

*Proof.* Part (i) follows by taking $E = L_p$ in the above theorem. Now assume (ii) holds. Then $F' = (M(L_p, F) \cdot L_p)' = M(M(F', L_{p'}), L_{p'})$ is $p'$-convex, and thus $F$ is $p$-concave. $\square$

As an application of the previous theorems we present a corollary, which reproves Theorem 1 of [17]. We shall only present the isometric case.

**Corollary 3.10.** *Let $1 < p < q < \infty$ and let $s$ be defined by $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$. Then the following are equivalent.*

 (i) *$E$ is $p$-convex and $q$-concave with convexity and concavity constants equal to 1.*
 (ii) *$M(L_q, E) \cdot M(E, L_p)$ is a product Banach function space and $L_s = M(L_q, E) \cdot M(E, L_p)$.*

*Proof.* Assume (i) holds. Then $E$ is $q$-concave for $q < \infty$, so $E$ has the Fatou property (see [8]). Hence by the above theorem we have that $L_q \cdot M(L_q, E) = E$. This implies that $L_q^{\frac{1}{2}} \cdot M(L_q, E)^{\frac{1}{2}} \cdot M(E, L_p)^{\frac{1}{2}} = E^{\frac{1}{2}} \cdot M(E, L_p)^{\frac{1}{2}} = L_p^{\frac{1}{2}}$, since $E$ is $p$-convex. On the other hand also by Hölder's inequality $L_q^{\frac{1}{2}} \cdot L_s^{\frac{1}{2}} = L_p^{\frac{1}{2}}$. Hence $M(L_q, E)^{\frac{1}{2}} \cdot M(E, L_p)^{\frac{1}{2}} = L_s^{\frac{1}{2}} = L_{2s}$ by Corollary 2.6. This implies that $M(L_q, E)^{\frac{1}{2}} \cdot M(E, L_p)^{\frac{1}{2}}$ is 2-convex, so that $M(L_q, E) \cdot M(E, L_p)$ is a product

Banach function space and $L_s = M(L_q, E) \cdot M(E, L_p)$. Now assume (ii) holds. We will use then that $L_q \cdot L_{p'} = L_{s'}$ to get that

$$L_2 = L_s^{\frac{1}{2}} L_{s'}^{\frac{1}{2}} = (L_q \cdot M(L_q, E))^{\frac{1}{2}} \cdot (L_{p'} \cdot M(E, L_p))^{\frac{1}{2}} \subset E^{\frac{1}{2}} \cdot (E')^{\frac{1}{2}} = L_2.$$

It follows now from Proposition 2.7 that $L_q \cdot M(L_q, E) = E$ and $L_{p'} \cdot M(L_{p'}, E') = E'$. This implies that $E$ is $q$-concave and $E'$ is $p'$-concave. As $E$ is $q$-concave for $q < \infty$, $E$ has the Fatou property and it follows that $E = E''$ is $p$-convex. $\square$

## 4. Applications

In [1] G. Bennett showed that many classical inequalities involving the $\ell_p$-norm can be expressed as product Banach sequence spaces. We will show that some of his results are easy consequences of the results of the previous sections. We start with his result about the spaces $d(\mathbf{a}, p)$ and $g(\mathbf{a}, p)$. Let $\mathbf{a} = (a_1, a_2, \cdots)$ be a non-negative sequence of real numbers with $a_1 > 0$. Define $A_n = a_1 + \cdots a_n$. Then for $p \geq 1$ the Banach sequence spaces $d(\mathbf{a}, p)$ and $g(\mathbf{a}, p)$ are defined as follows:

$$d(\mathbf{a}, p) = \{\mathbf{x} : \|\mathbf{x}\|_{d(\mathbf{a,p})} < \infty\},$$

where

$$\|\mathbf{x}\|_{d(\mathbf{a},p)} = \left( \sum_{n=1}^{\infty} a_n \sup_{k \geq n} |x_k|^p \right)^{\frac{1}{p}}$$

and

$$g(\mathbf{a}, p) = \{\mathbf{x} : \|\mathbf{x}\|_{g(\mathbf{a,p})} < \infty\},$$

where

$$\|\mathbf{x}\|_{g(\mathbf{a},p)} = \sup_n \left( \frac{1}{A_n} \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}.$$

**Theorem 4.1.** *Let $1 \leq p < \infty$. Then $d(\mathbf{a}, p) \cdot g(\mathbf{a}, p)$ is a product Banach sequence space and $d(\mathbf{a}, p) \cdot g(\mathbf{a}, p) = \ell_p$.*

*Proof.* It suffices to prove the theorem for $p = 1$. The general result follows then by $p$-convexication. To prove the theorem for $p = 1$ we need only show, by Lozanovskii's factorization theorem, that $d(\mathbf{a}, 1)' = g(\mathbf{a}, 1)$. For the inequality $\|\mathbf{y}\|_{d(\mathbf{a},1)'} \leq \|\mathbf{y}\|_{g(\mathbf{a},1)}$ we refer to the first part of the proof of Theorem 3.8 of [1]. For the reverse inequality observe that

$$\frac{1}{A_n} \sum_{k=1}^{n} |y_k| \leq \|\frac{1}{A_n}(1, 1, \cdots, 1, 0, \cdots)\|_{d(\mathbf{a},1)} \|\mathbf{y}\|_{d(\mathbf{a},1)'} = \|\mathbf{y}\|_{d(\mathbf{a},1)'}$$

for all $n \geq 1$, which proves $\|\mathbf{y}\|_{d(\mathbf{a},1)'} = \|\mathbf{y}\|_{g(\mathbf{a},1)}$. $\square$

**Remark.** The above theorem reproves Theorem 3.8 of [1]. There the factorization was proved directly by a lengthy argument.

A direct consequence of Theorem 3.7 and the above theorem is the following theorem, which corresponds to Theorems 12.3 and 12.22 of [1], where again direct proofs were given.

**Theorem 4.2.** *Let $1 < p < \infty$. Then*

$$d(\mathbf{a}, p)' = \ell_{p'} \cdot g(\mathbf{a}, p),$$

*and*

$$g(\mathbf{a}, p)' = \ell_{p'} \cdot d(\mathbf{a}, p),$$

*where the right hand sides are product Banach sequence spaces.*

Next we will show how another theorem of [1] is a direct consequence of the results of the previous section. In this case we will present the result for arbitrary $\sigma$-finite measure spaces as that will allow us to consider arbitrary order continuous operators. Let $L \subset L_0(X, \mu)$ be an order ideal, which has the property that for each measurable set $A$ of positive measure contains a measurable subset $B$ of positive measure such that $\chi_B \in L$. Let $1 < p < \infty$ and $T : L \to L_p$ be a strictly positive order continuous linear map. Then there exists a maximal order ideal $\mathcal{D}_p \subset L_0$ such that $T(\mathcal{D}_p) \subset L_p$. Define $\|f\|_{\mathcal{D}_p} = \|T(|f|)\|_p$. Then it is straightforward to see that this defines a norm on $\mathcal{D}_p$ with the Fatou property. Hence $\mathcal{D}_p$ is a Banach function space with respect to the norm $\|\cdot\|_{\mathcal{D}_p}$.

**Theorem 4.3.** *The Banach function space $\mathcal{D}_p$ is $p$-concave with concavity constant equal to 1. Hence $L_p \cdot M(L_p, \mathcal{D}_p)$ is a product Banach function space and $\mathcal{D}_p = L_p \cdot M(L_p, \mathcal{D}_p)$.*

*Proof.* Let $0 \le f_1, \cdots, f_n \in \mathcal{D}_p$. Let $0 \le \alpha_k$ such that $\sum_{k=1}^{n} \alpha_k^{p'} \le 1$. Then have

$$\sum_{k=1}^{n} \alpha_k T f_k = T\left(\sum_{k=1}^{n} \alpha_k f_k\right) \le T\left(\left(\sum_{k=1}^{n} f_k^p\right)^{\frac{1}{p}}\right) \quad \text{a.e.}$$

By taking then the supremum over a countable dense set of the positive unit ball of $\ell_{p'}(n)$ we get that

$$\left(\sum_{k=1}^{n} (T f_k)^p\right)^{\frac{1}{p}} \le T\left(\left(\sum_{k=1}^{n} f_k^p\right)^{\frac{1}{p}}\right) \quad \text{a.e.}$$

Taking $L_p$-norms on both sides we get

$$\left(\sum_{k=1}^{n} \|f_k\|_{\mathcal{D}_k}\right)^{\frac{1}{p}} \le \left\|\left(\sum_{k=1}^{n} f_k^p\right)^{\frac{1}{p}}\right\|_{\mathcal{D}_p},$$

i.e., $\mathcal{D}_p$ is $p$-concave with concavity constant equal to 1. The remaining statements follow now from Theorem 3.9

$\square$

In [1] the factorization in the above theorem was proved for Banach sequence spaces as part of Theorem 17.6 by a completely different method, using Maurey's factorization theorem. With essentially the same argument as used above we can extend the above theorem by replacing $L_p$ by a $p$-concave Banach function with concavity constant equal to 1. We leave the details to the reader.

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